

Probabilistic voting equilibria under alternative candidate payoff functions.

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Abstract

In this paper I analyze the equilibrium in a probabilistic voting model where the candidates have preferences other than the maximization of the expected number of votes or the probability of win maximization. I derive the comparative statics for two voters and one-dimensional policy space. Each voter cares about both the policy platform and the identity of the candidate. It is shown that an increase in the value of exactly one vote causes each candidate to choose a position closer to that of its partisan voter. Numeric computation of equilibria show that these results can be generalized to three or more voters.

1 Introduction

Probabilistic voting models evolved as a response to the nonexistence of equilibrium in games of electoral competition with a multi-dimensional policy space. Such models assume that a voter's decision is a random variable that is continuous in the policy position of each candidate. This setting guarantees the existence of a mixed-strategy equilibrium, and, under broad conditions, of a local Nash equilibrium. The conditions for the existence of a global Nash equilibrium are, however, much less transparent, as they require concavity of candidate objective functions (Hinich, Ledyard, and Ordeshook, 1972,1973, Hinich, 1978, Linbeck and Weibull, 1987, Coughlin, 1992, Banks and Duggan, 2005). An additional benefit of a probabilistic model is that it

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can be integrated with an econometric multinomial choice model that can be used to estimate the individual probability of vote functions from mass survey data (Schofield 2007).

A persistent result in this literature was the “mean voter theorem” — the existence of an equilibrium where both (or all) candidates choose an identical policy position. Under the assumptions that the voters are identical with respect to their nonpolicy preferences toward the candidates, the equilibrium position maximizes the total expected utility of the voters. In a two-candidate setting, it was shown that under much broader conditions the only equilibrium that can exist is the one in which the policy positions of the candidates are identical (Banks and Duggan, 2005). Numerical methods were used to demonstrate that other, nonsymmetric local equilibria may exist in models with several political parties or candidates (Quinn and Martin, 2002, Schofield and Sened, 2006). However, the properties of such divergent equilibria are known only to a limited degree. In particular, Schofield (2007) has shown that in a multi-candidate probabilistic voting game with multi-dimensional policy space, the positions of all candidates are located on one line.¹ Yet it would be safe to say that probabilistic voting models (especially the two-candidate ones) were either unable to produce an equilibrium with nonidentical policy platforms, or were unable to explain the comparative statics of the equilibrium.

This work uses the probabilistic voter environment to analyze the equilibrium effects of the previously unexplored factor — candidate preferences with respect to the number of votes received in an election. Consider, for example, an election with N voters and two candidates. There are $N + 1$ possible election results: Candidate 1 getting 0 votes, Candidate 1 getting 1 vote, and so on. The payoff that each candidate receives with each result depends on several factors. In a pure winner-take-all election, the payoff is zero if the number of votes is a strict minority, and one if it is a strict majority. However, this payoff function is distorted if there exists a consolation prize to a candidate who loses by a narrow margin, or if winning by a large margin confers additional benefits such as legitimacy, security, or a popular mandate for reforms. If we consider competition between political parties, the relationship between votes and payoffs depends on the election system proportionality as well as on how well seats in the legislature can be converted into policy or office benefits.

The bulk of the probabilistic models assumed political agents that maximized the expected number of votes, which corresponds to a linear function that translates votes into payoffs. The remaining works assumed winner-take-all payoffs. The equivalence of candidate behavior un-

¹The same work gave the conditions under which the convergent equilibrium would unravel due to failure of second-order conditions.

der these two assumptions attracted the attention of several scholars. Hinich (1977), Ledyard (1984) and Duggan (2000) argued in favor of the strategic equivalence of these two assumptions under Euclidian voter preferences and additive uncertainty. However, Patty (2005, 2007) demonstrated that under more general assumptions about the probability of voting functions, the response functions of probability-of-victory maximizers are different from those of expected vote share maximizers, unless some very special conditions on the voting probabilities are met.

In a recent work, Zakharov (2009) has shown that in a two-candidate probabilistic voting model, policy convergence occurs if and only if a strict symmetry condition on the payoff functions are satisfied. Both expected utility maximizers and probability of win maximizers are special cases that satisfy this condition. Most other functions, including all concave functions, do not.

This paper is a continuation of my previous work. While Zakharov (2009) was a (non)existence result, here I derive the comparative statics of the equilibrium. This is done analytically for the simplest two-voter case; this result is augmented by a numeric calculation of Nash equilibria for several voters.

2 Main results

There are 2 candidates who compete in an election by choosing policy platforms $y_1, y_2 \in [0, 1]$. There are two voters, 1 and 2. Let $P_i(y_1, y_2)$ be the probability that voter $i = 1, 2$ votes for Candidate 1, and $1 - P_i(y_1, y_2)$ the probability that she votes for Candidate 2. Suppose that the votes are independent. Assume that the voters behave according to the utility-difference model:

$$P_i(y_1, y_2) = P(u_{i1} - u_{i2}), \quad (1)$$

where u_{ij} is the utility that voter i attributes to Candidate $j = 1, 2$, and $P(\cdot)$ is a continuous, differentiable, strictly increasing function. Let

$$u_{ij} = e_{ij} - \psi(y_j - v_i), \quad (2)$$

where e_{ij} is the nonpolicy preference of voter i for Candidate j , $v_i \in [0, 1]$ is the best policy of voter i , and $\psi(\cdot)$ is a twice-differentiable disutility function that is symmetric around 0, with $\psi'(0) = 0$, $\psi'(d) > 0$ for $d > 0$, and $\psi''(d) > 0$. Let $v_1 = 0$ and $v_2 = 1$. Without loss of generality, let $e_{12} = e_{21} = 0$.

The payoff of each candidate depends on the number of votes he receives. There are 3 possible election results: 2 votes going to Candidate 1, 1 vote for each candidate, and 2 votes for Candidate 2. Assume that each candidate's payoff is a nondecreasing function of the number of votes. Then, without loss of generality, let the utility of 0 votes be 0, the utility of 2 votes be 1, and the utility of 1 vote be $x \in [0, 1]$.

A high x implies that a candidate values winning one half of all votes relatively high compared to winning all votes, or no votes at all. There are several factors that can affect the value of this parameter. For example, suppose that in the event of a 50-50 vote split, the election outcome is decided by a coin toss. If there is a high consolation prize to the losing candidate, x will be higher. On the other hand, if the winner needs a clear majority mandate, x will be lower.

The expected utility functions for both candidates will be

$$U_1 = x((1 - P_1)P_2 + P_1(1 - P_2)) + P_1P_2, \quad (3)$$

$$U_2 = x((1 - P_1)P_2 + P_1(1 - P_2)) + (1 - P_1)(1 - P_2). \quad (4)$$

For $x = \frac{1}{2}$ the utilities are equal to the expected share of the total vote: $U_1 = \frac{1}{2}P_1 + \frac{1}{2}P_2$, $U_2 = 1 - \frac{1}{2}P_1 - \frac{1}{2}P_2$. This special case was analyzed in most of the previous literature.

I now formulate the main analytic results of this work. Local Nash equilibrium (LNE) will be used as the solution concept.

Proposition 1 *Suppose that $e_{11} = e_{22} = e$. Let $P(x) = 1 - P(-x)$. Then there exists a local equilibrium in the electoral competition game with $y = y_1 = 1 - y_2$. The equilibrium is given by*

$$\psi'(y)(P + x - 2Px - 1) + \psi'(1 - y)(x + P - 2Px) = 0. \quad (5)$$

In this setting there are two voters. The value e can be interpreted as a voter's degree of partisanship, or the degree to which a voter supports "her" candidate if the policy positions of the two candidates are identical. Thus, if there are two voters whose probability of vote functions are identical up to the ideal points and the identity of the preferred candidate, then we should expect the candidates to choose policy positions that are symmetric with respect to the voter ideal points.

The comparative statics of the equilibrium are summarized in the following proposition.

Proposition 2 *Suppose that $(y, 1 - y)$ is a symmetric equilibrium in the electoral competition game. Then y decreases with x for $x \leq \frac{1}{2}$ and y increases with e for $x < \frac{1}{2}$. Suppose also that*

$$P'(e - \psi(y) + \psi(1 - y))(\psi'(y) + \psi'(1 - y))^3 < \psi'(y)\psi''(1 - y) + \psi'(1 - y)\psi''(y) \quad (6)$$

for all $y < \frac{1}{2}$. Then y decreases with x for all $x \in [0, 1]$. Also, y increases with e for $x < \frac{1}{2}$ and decreases with e for $x > \frac{1}{2}$.

It follows that, as the value of x increases, we should expect the candidates to choose policy positions closer to the ideal points of their partisan voters; the magnitude of this effect is increasing in the degree of partisanship. Indeed, suppose that the voters are highly partisan: Voter 1 will vote for her preferred candidate (Candidate 1) unless the policy position of Candidate 2 is much closer to her ideal point. Thus, the utility of both candidates is high if they both locate close to the ideal points of their partisan voters.

Interestingly, the model predicts that if the candidates are risk-loving, each will choose a position that is closer to the ideal policy of the *opposing* candidate's partisan voter. This is true because y is monotonic in x , and because for $x = \frac{1}{2}$ the mean-voter theorem holds, so $y = \frac{1}{2}$. Suppose that both voters are partisan, and the candidates are risk-lovers. In order to maximize the probability of winning both votes, Candidate 1 must maximize the probability of Voter 2 supporting him (given that the probability of Voter 1 supporting Candidate 1 is high enough). So, he will locate near Voter 2.

Example. The disutility functions be as follows:

$$u_{ij} = e_{ij} - \beta|v_i - y_j|^\delta, \quad (7)$$

where v_i is the ideal policy of Voter i , and $\delta \geq 1$. Suppose that

$$P(u_1 - u_2) = \begin{cases} 0, & u_1 - u_2 < -\alpha \\ \frac{\alpha + u_1 - u_2}{2\alpha}, & u_1 - u_2 \in [-\alpha, \alpha] \\ 1, & u_1 - u_2 > \alpha, \end{cases} \quad (8)$$

with $\alpha \geq \beta + e$. Then for $\delta = 2$ the equilibrium equation (5) can be rewritten as

$$y = \frac{\alpha + \beta + e - 2x\beta - 2xe}{2\alpha - 2\beta + 4x\beta}. \quad (9)$$

We have, for all e and β and $\frac{\partial y}{\partial x} < 0$ and $\frac{\partial^2 y}{\partial e \partial y} < 0$. The sign of $\frac{\partial y}{\partial x}$ is positive for $x < \frac{1}{2}$ and negative for $x > \frac{1}{2}$.

Example. Let the disutility be given as in (7). The equilibria for the symmetric two-voter model, as well as for a more general setting with a greater number of voters, can be calculated using numeric methods. The local equilibrium was found using a gradient hill-climbing algorithm implemented in Matlab. The results obtained numerically augment those obtained by analytical methods.

I used the logistic probability of vote function

$$P(u_1 - u_2) = \frac{e^{u_1}}{e^{u_1} + e^{u_2}} = \frac{e^{u_1 - u_2}}{e^{u_1 - u_2} + 1}. \quad (10)$$

Figure 1 shows the calculated equilibrium for different values of x and e .

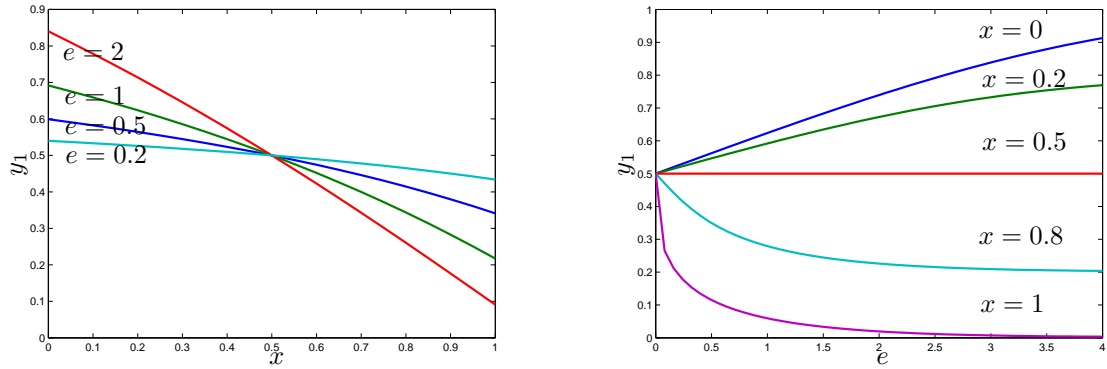


Figure 1: The equilibrium position of Candidate 1 for different values of x and e , with $\beta = 0.5$.

One can see that, for the chosen probability of vote and disutility functions, y is monotonic with respect to x for all values of x , for all ranges of e and β . Qualitatively, the relationship between e , x and y does not change if one chooses a different δ ; in particular, I looked at $1.1 < \delta \leq 4$ in 0.1 increments.

Example. Again consider the power loss function (7). Let the number of voters be $n > 0$. Denote by P_i the probability that voter i supports Candidate 1. Let V_j be the number of votes received by Candidate $j = 1, 2$. The probability that Candidate 1 receives exactly l votes is

$$p(V_1 = l) = \sum_{S \subseteq N, |S|=l} \left(\prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i) \right), \quad (11)$$

where $N = \{1, \dots, n\}$.

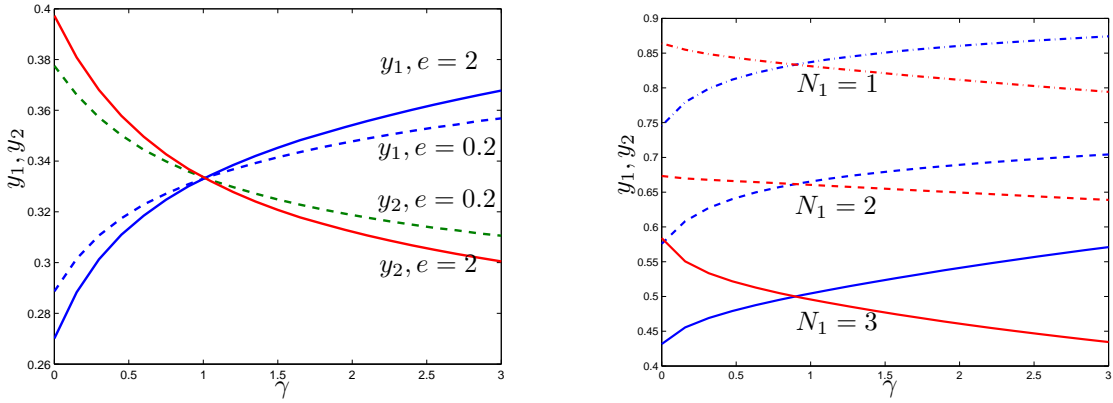
I assume that the candidates have the Cobb-Douglas utility function over the number of votes:

$$U_j = V_j^{\gamma_j}, \quad (12)$$

where V_j is the number of votes in favor of Candidate j , and $\gamma_j \geq 0$ is the parameter that determines the risk preference of the candidate. If $\gamma_j \in [0, 1)$, then the candidate are risk-averse; if $\gamma_j = 1$, he is risk-neutral; finally, if $\gamma_j > 1$, then candidate j is a risk-lover.

In order to obtain equilibrium comparative statics with respect to γ_j and other parameters, I calculated equilibrium policy positions for various values of model parameters. I assumed that there are two groups of voters of sizes $N_1 + N_2 = N$. For voter j in Group 1, took $v_j = 0$, $e_{1j} = e$, and $e_{2j} = 0$. For voter j in Group 2, I had $v_j = 1$, $e_{1j} = 0$ and $e_{2j} = e$. I fixed $\beta = 1$.

The comparative statics results are similar to the two-voter analytic results. Figure 2(a) shows equilibrium positions of the two candidates for the case when there are three voters, with $N_1 = 2$. The candidates had identical utility functions: $\gamma_1 = \gamma_2 = \gamma$.



(a) Equilibrium policy positions, three voters (b) Equilibrium policy positions, six voters, $e = 1$

Figure 2(b) shows the effect of the number of partisan voters on the equilibrium. As the number of partisans for Candidate 1 (the size of Group 1) decreases, the equilibrium positions of both candidates move closer to the ideal policy of Candidate 2's partisans. In all cases, increases in γ cause candidates to adopt policy positions that are further away from the ideal policy position of their partisan voters; for $\gamma = 1$ the candidates choose identical policy positions, as predicted by theory (Banks and Duggan, 2005).

As the number of voters increases, the policy positions of the candidates coincide when both candidates are risk-neutral. Table 1 shows the calculated equilibria for several scenarios when the number of voters N and number of Candidate 1 partisans are multiples of 3 and 2, respectively.

Table 2 shows similar results for voter groups of equal size, and candidates with different utility functions. It still appears that the equilibrium policy positions converge to those for risk-neutral candidates: $y_1 = y_2 = 0.5$.

		$N = 3$	$N = 6$	$N = 9$	$N = 12$	$N = 15$
		$N_1 = 2$	$N_1 = 4$	$N_1 = 6$	$N_1 = 8$	$N_1 = 10$
$\gamma_1 = 0.2$	y_1	0.4652	0.4157	0.3714	0.3540	0.3476
$\gamma_2 = 0.2$	y_2	0.2151	0.2903	0.3116	0.3186	0.3221
$\gamma_1 = 0.5$		0.4031	0.3658	0.3500	0.3443	0.3415
$\gamma_2 = 0.5$		0.2762	0.3108	0.3200	0.3237	0.3264
$\gamma_1 = 1$		0.3333	0.3333	0.3333	0.3333	0.3333
$\gamma_2 = 1$		0.3333	0.3333	0.3333	0.3333	0.3333
$\gamma_1 = 2$		0.2665	0.2992	0.3102	0.3160	0.3194
$\gamma_2 = 2$		0.4056	0.3686	0.3567	0.3508	0.3472

Table 1: Equilibrium policy positions.

		$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 12$
		$N_1 = 2$	$N_1 = 3$	$N_1 = 4$	$N_1 = 5$	$N_1 = 6$
$\gamma_1 = 0.5$	y_1	0.5519	0.5286	0.5183	0.5135	0.5107
$\gamma_2 = 1$	y_2	0.5006	0.5002	0.5001	0.5000	0.5000
$\gamma_1 = 0.5$		0.5502	0.5276	0.5179	0.5132	0.5105
$\gamma_2 = 2$		0.5577	0.5385	0.5289	0.5231	0.5193
$\gamma_1 = 1$		0.4992	0.4996	0.4998	0.4999	0.4999
$\gamma_2 = 2$		0.5583	0.5387	0.5290	0.5232	0.5194

Table 2: Equilibrium policy positions for groups of voters of equal size.

The intuition behind this convergence of candidate policy positions is straightforward. If the voting is probabilistic, then the vote share of each candidate is a random variable. If the votes are not correlated, then this random variable converges to the candidate's expected vote share as the number of voters increases. Hence the candidates act as expected vote share maximizers if the number of voters is large.

The following proposition investigates the equilibrium comparative statics with respect to the properties of the voter disutility functions.

Proposition 3 *Let $\psi_k(\cdot)$ be a family of twice continuously differentiable disutility functions indexed by $k = 1, 2, \dots$ such that for all k we have $\psi'_k(x) = \psi'_k(-x)$, $\psi'(x) > 0$ for $x > 0$, $\psi'_k(0) = 0$, and*

$$\lim_{k \rightarrow \infty} \psi'_k(x) = \beta \quad (13)$$

for all $x > 0$. Then there exists a sequence of equilibria $(y_k, 1 - y_k)$ such that

$$\lim_{k \rightarrow \infty} y_k = 0 \quad (14)$$

for $x > \frac{1}{2}$ and

$$\lim_{k \rightarrow \infty} y_k = 1. \quad (15)$$

for $x < \frac{1}{2}$.

It follows that as the disutility functions become more and more linear, the policy divergence becomes greater, except in the knife-edge case $x = \frac{1}{2}$.

Example. Let the voter disutility functions be linear in policy distance: $\delta = 1$. In that case, for any $y_1 = 1 - y_2$ we are going to have $\frac{\partial U_1}{\partial y_1} < 0$ and $\frac{\partial U_2}{\partial y_2} > 0$ for $x > \frac{1}{2}$ and the opposite signs for $x < \frac{1}{2}$. Hence, there will be a corner equilibrium $y_1 = 0, y_2 = 1$ for $x > \frac{1}{2}$ and $y_1 = 1, y_2 = 0$ for $x < \frac{1}{2}$.

Indeed, fix $y_2 = 1 - y_1$. Suppose that the position of Candidate 1 changes a little. As we assume that the probability of voting function $P(\cdot)$ are symmetric, and that the marginal policy disutility is constant, it follows that the marginal effects of this policy change on P_1 and P_2 will be equal. Thus the sign of the effect on U_1 will depend only on the sign of x , not on y_1 .

3 Conclusion

The results of this work suggest that a model of electoral competition with sincere, probabilistic voters and vote-maximizing candidates can do a better job at explaining observed electoral strategies of candidates and political parties than previously thought. The mean-voter equilibrium, for a two-candidate model, in fact depends on the assumption that the payoffs to the candidates are symmetric around 50% of the vote (Zakharov, 2009). A number of factors that introduce asymmetries, such as consolation prizes for well-performing losers or the need for a broad mandate, can affect the policy positions in a predictable manner. In general, we should expect that factors that cause the candidates to become more risk-averse will lead them to adopt policy positions closer to those of their partisan voters.

The numerical results suggest that, regardless of their preferences, the candidates will act as expected vote share maximizers if the number of voters is large. One is tempted to conclude that policy divergence due to risk nonneutrality should not be observed in any realistic election; however, this result crucially depends on the assumption that the votes are not correlated. In case there is a random shock common to all voter's preferences, then candidate payoffs will not be deterministic even in large elections. The effect of correlated voting on electoral equilibria is a subject for future theoretical and empirical research.

Proofs

Proof of Proposition 1. The first-order conditions for a Nash equilibrium are

$$\frac{\partial U_1}{\partial y_1} = x(P_{11} + P_{21}) + (1 - 2x)(P_{11}P_2 + P_{21}P_1) = 0, \quad (16)$$

$$\frac{\partial U_2}{\partial y_2} = (x - 1)(P_{12} + P_{22}) + (1 - 2x)(P_{12}P_2 + P_{22}P_1) = 0. \quad (17)$$

Note that for $y_1 = y$, $y_2 = 1 - y$, these conditions are coincidental, and correspond to a single condition

$$\psi'(y)(p + x - 2px - 1) + \psi'(1 - y)(x + p - 2px) = 0, \quad (18)$$

where $p = P(e - \psi(y) + \psi(1 - y))$. The sufficient condition that will guarantee us a Nash equilibrium for some $y_1 = y, y_2 = 1 - y$ is $\frac{\partial U_1}{\partial y_1} > 0$ for $y = 0$. Indeed, this condition is equivalent to $\frac{\partial U_1}{\partial y_1} < 0$ for $y = 1$. As $\frac{\partial U_1}{\partial y_1}$ is continuous in y , for some $y_1 = y, y_2 = 1 - y$ we must have a local maximum of U_1 . The proof is complete.

Proof of Proposition 2. Take derivatives of (25):

$$\begin{aligned} \frac{\partial H}{\partial y} &= -(1-2x)(\psi'(y) + \psi'(1-y))^2 p' + \\ &\quad + \psi''(y)(p+x-2px-1) - \psi''(1-y)(x+p-2px) \end{aligned} \quad (19)$$

$$\frac{\partial H}{\partial x} = (\psi'(1-y) + \psi'(y))(1-2p), \quad (20)$$

$$\frac{\partial H}{\partial e} = (1-2x)p'(\psi'(y) + \psi'(1-y)), \quad (21)$$

where $p' = P'(e - \psi(y) + \psi(1-y))$. We have $p > \frac{1}{2}$ for $y \leq \frac{1}{2}$ by definition. For $y > \frac{1}{2}$ we also have $p > \frac{1}{2}$ because of (25). Hence, $\frac{\partial H}{\partial x} < 0$ and

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial y}} < 0 \quad (22)$$

whenever $\frac{\partial H}{\partial y} < 0$.

The sign of $\frac{\partial H}{\partial e}$ is equal to the sign of $1-2x$. Hence

$$\frac{\partial y}{\partial e} = -\frac{\frac{\partial H}{\partial e}}{\frac{\partial H}{\partial y}} \quad (23)$$

is positive if $x < \frac{1}{2}$ and negative if $x > \frac{1}{2}$, given $\frac{\partial H}{\partial y} < 0$.

We have $\frac{\partial H}{\partial y} < 0$ if $x \leq \frac{1}{2}$. If $x > \frac{1}{2}$, the sufficient condition that guarantees $\frac{\partial H}{\partial y} < 0$ is

$$p'(\psi'(y) + \psi'(1-y))^3 < \psi'(y)\psi''(1-y) + \psi'(1-y)\psi''(y) \quad (24)$$

for all $y < \frac{1}{2}$. The proof is complete.

Proof of Proposition 3. Take $y > 0$ and $x > \frac{1}{2}$. There exists $k(y)$ such that for all $k > k(y)$ we have

$$\psi'_k(y)(p+x-2px-1) + \psi'_k(1-y)(x+p-2px) < 0, \quad (25)$$

as for any $\epsilon > 0$ we have $|\psi'_k(y) - \psi_k(1-y)| < \epsilon$ for a high enough k . As $\frac{\partial U_1}{\partial y_1} > 0$ for $y_1 = 0$ and $y_2 = 1$, there exists a local equilibrium $(y', 1-y')$ with $0 < y' < y$. Similarly, one can show that for $x < \frac{1}{2}$, y converges to 1 as $k \rightarrow \infty$. The proof is complete.

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