Policy convergence in a two-candidate probabilistic voting model.

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Abstract

I propose a generalization of the probabilistic voting model in two-candidate elections. Unlike in all previous works, I assume that the candidates have general von Neumann-Morgenstern utility functions defined over the voting outcomes. For a finite number of voters, I derive necessary and sufficient conditions for the existence of a local Nash equilibrium in which the policy platforms of the candidates are identical. The convergent equilibrium exists only if a certain condition on the probability of voting functions is satisfied, and/or a strict symmetry condition on the candidate utility functions is satisfied. Both widely studied special cases — probability of win-maximizers and voteshare-maximizers — satisfy the latter condition. Hence, policy coincidence will not exist in the general case, although Banks and Duggan (2005) have shown that every pure-strategy Nash equilibrium will exhibit convergence for voteshare-maximizing candidates and a broad class of probability of voting functions.

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1 Introduction

A well-known result in the spatial voting theory is the mean voter theorem, due to Hinich, Ledyard, and Ordeshook (1972) and Hinich (1977,1978). In a nutshell, the theorem's statement is as follows. Supppose that the probability of a voter supporting a party (or candidate) depends on the difference between the utilities that he attributes to the two parties. Suppose that the marginal effect of an increase in the utility difference on the probability of voting is equal across voters. Then, if the candidates maximize expected voteshare, the first-order conditions for Nash equilibrium are met if the two candidates select identical policy positions. Moreover, that policy position should maximize the sum of voter utilities.

The early works considered the voters to have quadratic disutility from policy distance. Under such assumptions, the convergent equilibrium is located at the mean of the voter ideal policies. Recent generalizations include generalized form of Eucledian preferences in Lin, Enelow, and Dorussen (1999), and strategic voting behavior in McKelvey and Patty (2006). In all works a convergent equilibrium was shown to exist in the general case[1]. Schofield (2007) looked at probabilistic voting with quadratic disutility when the multiple candidates have different level of nonpolicy characteristics (valence), which is shown to affect the stability of the convergent equilibrium, but not its location or the fact that the first-order conditions are satisfied at the mean of the voter ideal policies.

Most works utilizing probabilistic voting models assume that the candidates are expected voteshare maximizers, or probability of victory maximizers. The equivalence of candidate behavior under these two assumptions attracted attention of several scholars. Hinich (1977), Ledyard (1984) and Duggan (2000) argued in favor of the strategic equivalence of these two assumptions under Eucledian voter preferences and additive uncertainty. However, Patty (2005, 2007) demonstrated that under more general assumptions about the probability of voting functions, the response functions of probability-of-victory maximizers are different from those of expected voteshare maximizers, unless some very special

[1]In deterministic multicandidate elections with strategic voters, policy divergence and multiple equilibria usually arise, as Patty, Snyder, and Ting (2008) demonstrate.

conditions on the voting probabilities are met.

The paper to which I compare most of my results is Banks and Duggan (2005), where the candidates are restricted to be voteshare maximizers, but there is a continuum of voters. In my work, I allow for general candidate utility functions.

I analyze a class of probability of voting functions that satisfies two conditions. Firstly, for each voter, a small change in the position of Candidate 1 must have the same effect as an equal, but opposite change in the position of Candidate 2. Secondly, the preferences must be satiated. Roughly, that means that the probability of a voter supporting a candidate must be small if the candidate's policy platform is too extreme. These conditions are satisfied, for example, by the probability of vote functions that are induced by a utility function with an additive bias that is equal across the voters (such as in Hinich (1987) and later works).

Two conditions, at least one of which must be satisfied for a convergent equilibrium to exist, are derived. The first condition is on the probability of voting functions: the probability of voting for Candidate 1 must be equal across voters, if the candidate positions are identical. The second condition is on the candidate utility functions. It is rather restrictive. In particular, if we assume that there are three voters and the preferences of the two candidates are identical, then both candidates must be indifferent between fair lotteries offering 0 or 3 votes, or 1 and 2 votes, respectively.

However, most types of political agents considered in the previous literature — like the expected utility maximizers, the probability of victory maximizers, and the plurality maximizers — are special cases that satisfy this exact condition. Thus I show that mean-voter convergence is an artifact of the assumptions about the candidate expected utility functons.

2 The model and the results

There are 2 candidates and N stochastic voters. The candidates engage in a one-shot game by choosing policy platforms y_j from a compact policy space $X \subset \mathbf{R}^k$, $j = 1, 2$. The voters have policy preferences manifested in the twice continuously differentiable probability of voting functions $p_i : X^2 \to (0,1)$, where $p_i(y_1, y_2)$ is the probability that voter i will support Candidate 1 given policy platforms y_1 and y_2. For any given y_1, y_2, the vote of voter i is independent of the votes of all other voters.

The payoff of a candidate depends on the number of votes she receives. Each candidate is endowed with a von Neumann-Morgenstern utility function. The expected utilities of the candidates are

$$U_1 = \sum_{l=0}^{N} P_l u_l^1, \tag{1}$$

$$U_2 = \sum_{l=0}^{N} P_l u_{N-l}^2, \tag{2}$$

where

$$P_l = \sum_{S \subseteq N, |S|=l} \left(\prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i) \right) \tag{3}$$

is the probability that Candidate 1 obtains exactly l votes, and u_l^j is the corresponding payoff to candidate j. It is assumed that $u_0^j \leq u_1^j \leq \cdots \leq u_N^j$, with $u_0^j < u_N^j$.

If y_1 maximizes $\sum_{l=0}^{N} P_l u_l^1$, then it also maximizes $\sum_{l=1}^{N} p_l(bu_l^1 + a) = b\sum_{l=1}^{N} p_l u_l^1 + a$ for any a and $b > 0$. Hence we can take $u_0^j = 0$ and $u_N^j = 1$, $j = 1, 2$, without loss of generality.

It is straightforward to show that an expected voteshare maximizer with $U_1 = \sum_{l=1}^{N} p_l$ will have $u_l^1 = \frac{l}{N}$. A probability of victory maximizer will have $u_l^1 = 0$ if $l < \frac{N}{2}$, $u_l^1 = \frac{1}{2}$ if $l = \frac{N}{2}$, and $u_l^1 = 1$ if $l > \frac{N}{2}$. An expected plurality maximizer will also have $u_l^1 = \frac{l}{N}$.

I now define the solution concept that I am going to use.

Definition. (y_1^*, y_2^*) is a local Nash equilibrium in the 2-player game with the strategy set X and payoffs (1), (2) if there exists $\epsilon > 0$ such that for every $|y_1' - y_1^*| < \epsilon$, for every $|y_2' - y_2^*| < \epsilon$, we have $U_1(y_1^*, y_2^*) \geq U_1(y_1', y_2^*)$ and $U_2(y_1^*, y_2^*) \geq U_2(y_1^*, y_2')$. The equilibrium is *interior* if y_1^*, y_2^* lie in the interior of X. It is *convergent* if $y_1^* = y_2^*$.

The first-order conditions for a local equilibrium are

$$D_1(U_1) = \sum_{l=0}^{N} u_1^l D_1(P_l) = 0 \tag{4}$$

$$D_2(U_2) = \sum_{l=0}^{N} u_2^l D_2(P_{N-l}) = 0. \tag{5}$$

where D_j denotes the partial derivative with respect to candidate j's policy platform[2]. I restrict my attention to interior equilibria. I will operate with the following assumptions about probability of vote functions.

Definition. The voters are *neutral* if for all y_1, y_2, we have $p_i(y_1, y_2) = 1 - p_i(y_2, y_1)$.

Definition. The voters are *equally biased at convergent positions* if for all $y_1 = y_2$ we have $p_1(y_1, y_2) = p_2(y_1, y_2) = p_3(y_1, y_2)$.

Definition. A voter is *marginally neutral at convergent positions* if $D_1(p_i(y_1, y_2)) = -D_2(p_i(y_1, y_2))$ for all $y_1 = y_2$.

Definition. A voter i is *satiated* if for all $z \in X$ there exists $\bar{\alpha} > 0$, $\bar{\beta} < 0$ such that for every $r \in \mathbf{R}^k$, $\|r\| = 1$, we have $z + \bar{\alpha} r \in X$ and $r \cdot D_1(p_i(z + \alpha r, z + \alpha r)) < \bar{\beta}$ for all $\alpha > \bar{\alpha}$.

The satiating condition implies that if both candidates choose some common policy that is too distant from some fixed policy position, then the gradient of the probability of voting function with respect to either candidate's position will point toward that policy position. This property is illustrated on Figure 1.

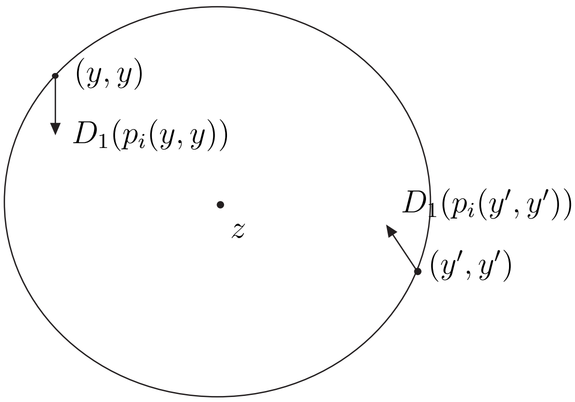


Figure 1: The preferences of voter i are satiated.

I will now relate the above definitions to some well-known conditions on the probability of voting functions that appear in Banks and Duggan (2005). The first example is the

[2] A weaker concept is the critical equilibrium (see Schofield and Sened, 2006), for which the first-order conditions are necessary and sufficient.

utility difference model, when the voter assigns a certain utility to each policy platform, and the probability of voting for candidate 1 depends on the difference between the utilities from the platforms of Candidates 1 and 2.

Example 1 *Suppose that p_i satisfy the* utility difference model*, namely,*

$$p_i(y_1, y_2) = \hat{p}_i(u_i(y_1) - u_i(y_2)), \tag{6}$$

where $\hat{p}_i(\cdot)$ is a non-decreasing differentiable function, and $u_i(\cdot)$ is a differentiable function.[3]

Then, i is neutral if and only if $\hat{p}_i(x) = 1 - \hat{p}_i(-x)$ for all x; voters are equally biased at convergent positions if $\hat{p}_i(0) = \hat{p}^0$ for all i; all voters are marginally neutral at convergent positions; voter i is satiated if and only if there exist $v_i \in X$, $\bar{\alpha}_i > 0$, and $\bar{\beta}_i < 0$ such that for all $r \in \mathbf{R}^k$, $\|r\| = 1$, we have $z + \bar{\alpha}_i r \in X$ and $r \cdot D(u_i(v_i + \alpha r)) < \bar{\beta}_i$ for all $\alpha > \bar{\alpha}_i$.

Hence for the utility difference model the satiability of $p_i(\cdot)$ is equivalent to the satiability of $u_i(\cdot)$. The second example is a special case of the utility difference model.

Example 2 *Suppose that p_i satisfy the conditions of the* probabilistic voting model with generalized Eucledian preferences*, namely, voter i supports candidate 1 if*

$$-(y_1 - v_i)A_i(y_1 - v_i)^T + \delta_i + \epsilon_i < -(y_2 - v_i)A_i(y_2 - v_i)^T, \tag{7}$$

where $v_i \in X$ is the ideal policy of voter i, A_i is a non-negative definite $n \times n$ matrix, ϵ_i is a zero-mean random variable, and δ_i is the bias of voter i in favor of candidate 1.

Then, i is neutral if and only if the density of each ϵ_i is symmetric around 0, and $\delta_i = 0$; voters are equally biased at convergent positions if and only if $\delta_i = 0$ for all i; all voters are marginally neutral at convergent positions and all voters are satiated.

The final example is when the probability of voting for candidate 1 depends on the ratio of the utilities from the platforms of candidates 1 and 2. This specification first appeared in Coughlin and Nitzan (1981).

[3]This definition is slightly different from Banks and Duggan (2005), as I assume no possibility of abstention for the voters.

Example 3 *Suppose that* p_i *satisfy the* utility ratio model*, namely,*

$$p_i(y_1, y_2) = \hat{p}_i(u_i(y_1)/u_i(y_2)), \tag{8}$$

where $\hat{p}_i(\cdot)$ *is a non-decreasing differentiable function, and* $u_i(\cdot) > 0$ *is a differentiable function.*

Then, i *is neutral if and only if* $\hat{p}_i(x) = 1 - \hat{p}_i(\frac{1}{x})$ *for all* $x \neq 0$*; voters are equally biased at convergent positions if* $\hat{p}_i(1) = \hat{p}^1$ *for all* i*; all voters are marginally neutral at convergent positions; voter* i *is satiated if and only if there exist* $v_i \in X$*,* $\bar{\alpha}_i > 0$*, and* $\bar{\beta}_i < 0$ *such that* $r \in \mathbf{R}^k$*,* $\|r\| = 1$*,* $r \cdot D(u_i(v_i + \alpha r)) < \bar{\beta}_i$ *for all* $\alpha > \bar{\alpha}_i$*.*

Neutrality and equal bias at convergent positions are strong assumptions that do not hold in most empirical estimations of voter utility functions. Estimations of voter utility functions of type (7), based on mass survey data, show that coefficients δ_i significantly depend on observable characteristics such as gender, income, occupation, or religion (Schofield (2007) and Schofield and Zakharov (2009)). Marginal neutrality at convergent positions, on the other hand, is a weaker assumption. In order for it to be satisfied, it is sufficient that a voter's marginal disutility from a change in a candidate's policy does not depend on the identity of the candidate.

What are the necessary conditions on u_l^js for the existence of a local Nash equilibrium? Consider the following symmetry condition:

$$u_l^1 = 1 - u_{N-l}^2 \text{ for all } l = 0, \ldots, N. \tag{9}$$

The following result is available:

Theorem 1 *Suppose that the voters are marginally neutral at convergent positions and are satiated. Then a convergent equilibrium exists if the voters are equally biased at convergent positions, and/or if (9) holds. If (9) fails, then a convergent equilibrium does not exist for almost all probability of voting functions that are not equally biased at convergent positions.*

A convergent equilibrium in a k-dimensional policy space requires the solution of $2k$ first-order conditions (4), (5) in k unknowns. If condition (9) is satisfied, the k equations

(5) become redundant. If the voters are equally biased at convergent positions, then it can be shown that both (4), (5) are satisfied when $\sum_i D_i(p_i) = 0$.

These conditions have an interpretation. Suppose that $L(p)$ is some lottery that offers $0 < l < N$ votes with probability $1 - p$ and $N - l$ votes with probability p, and $M(p')$ is a lottery that offers 0 votes with probability $1 - p'$ and N votes with probability p'. If (9) hold, then there exists x such that candidate 1 prefers L to $M(p')$ if and only if candidate 2 prefers L to $M(p' + x)$ for all p, p'.

If the utility functions of the two candidates are identical, then (9) implies they are symmetric: $u_l^l = 1 - u_{N-l}^1$ all $l = 0, \dots, N$. Stated verbally, both candidates must be indifferent between the fair lottery that offers 0 or N votes, and a fair lottery that offers l or $N - l$ votes for all $1 \leq l \leq \frac{N}{2}$.

Note that conditions (9) are satisfied if both candidates are voteshare maximizers, or if both candidates are victory probability maximizers. It follows that both common assumptions made about the objective functions of the candidates in two-candidate models in fact satisfy a knife-edge condition. A convergent equilibrium will not exist under almost all other objective functions. For example, (9) is violated if one candidate is a voteshare maximizer and the other is a victory probability maximizer.

The intuition behind the lack of a convergent equilibrium can be illstrated through a simple numeric example. Using a gradient search algorithm implemented with Matlab 7.0, I obtained local Nash equilibria for a familiy of problems, varying a parameter of candidate objective functions.

Suppose that $N = 3$. Voter preferences are Eucledian. Utilities of voter i due to both candidates are given by

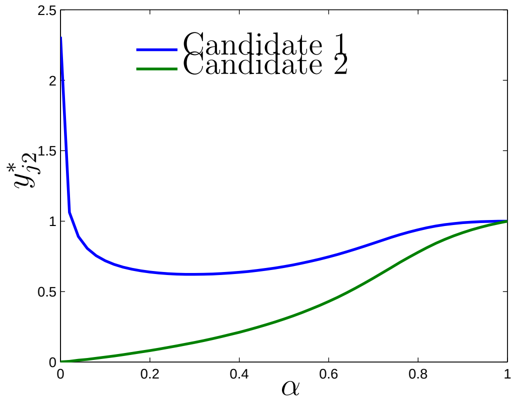
$$u_{i1} = \delta_i - \beta \|v_i - y_1\| + \epsilon_i, \quad (10)$$

$$u_{i2} = -\beta \|v_i - y_2\|, \quad (11)$$

where ϵ_i are independent and distributed logistically. Take $v_1 = (0, 3)$, $v_2 = (-1, 0)$, and $v_3 = (1, 0)$. Let $\beta = \frac{1}{2}$. Let $a_1 = 2$ and $a_2 = a_3 = -1$. It follows that, candidate policies being equal, Voter 1 prefers Candidate 1, while Voters 2 and 3 prefer Candidate 2.

Suppose that the utility functions of both candidates are identical, with $u_l^j = (\frac{l}{3})^\alpha$.

From numeric simulations it appears that equilibrium candidate platforms are given by $y_j^* = (0, y_{j2}^*)$, where y_{j2}^* are generally different and depend on α (see Figure 2).



For $\alpha = 1$, when candidates are risk-neutral, their positions are identical. As α decreases and candidates become more risk-averse, their policy positions move closer to the ideal policies of voters 2 and 3. As α approaches zero, Candidate 1 finds it safer to shift her platform closer to the ideal policy of voter 1, maximizing the probability of receiving at least one vote.

3 Discussion

One should compare this work's theorem with Theorem 6 of Banks and Duggan (2005), where the authors show that for voteshare maximizing candidates and utility difference, any pure-strategy Nash equilibrium must be convergent, as long as certain concavity conditions are satisfied. I show that a local pure-strategy Nash equilibrium exists for a narrow class of candidate utility functions under even some more general conditions on the probability of vote functions. The proof of my result is also non-constructive. However, for the general case of candidate utility functions, an additional condition is required — the voters must be equally biased toward the candidates if the policy platforms of the candidates coincide.

I show that if the candidate objective functions do not satisfy a symmetry condition, the convergent equilibrium unravels. Thus my work suggests an additional source of policy divergence in spatial voting models: objective functions of the candidates. In previous works, Schofield and Sened (2006) focused on the existence of political activists, Groseclose (2001) assumed partial policy motivation, and Zakharov (2008) looked at costly endogenous valence and strategic behavior by the candidates. The results of this work suggest that the cause of policy divergence may be more basic.

Also, my results demonstrate that an equilibrium where the candidates choose identical policy positions may exist under a broader range of conditions than previously thought. For example, my work shows that it is reasonable to expect a convergent equilibrium if the biases of the voters toward one of the candidates are different, as long as the candidates are voteshare maximizers or plurality maximizers, like in some previous works, such as Banks and Duggan (2005). In line with earlier results, one can show that in utility difference models, the total utility is maximized in a convergent equilibrium.

Appendix.

Proof of Theorem 1.

We have $D_1(P_l) = -D_2(P_l)$ whenever $y_1 = y_2$. Since $D_1(P_N) = -\sum_{l=0}^{N-1} D_1(P_l)$, from (5) we have

$$D_2(U_2) = D_1(P_N) + \sum_{l=0}^{N-1} (1 - u^2_{N-l}) D_1(P_l) = 0. \tag{12}$$

This equation is equivalent to (4) when (9) holds.

Now suppose that the voters are equally biased at convergent positions, so $p_l(y_1, y_2) = p$. We have

$$D_1(P_l) = \sum_{i=1}^{N} D_1(p_i) \times W_l = GW_l, \tag{13}$$

where

$$W_l = p^{l-1}(1-p)^{N-l-1}\left((1-p)C^{N-1}_{l-1} - pC^{N-1}_l\right). \tag{14}$$

As $D_1(p_i) = -D_2(p_i)$, the first order conditions can be rewritten as

$$D_1(U_1) \;=\; G\sum_{l=0}^{N} u^1_l W_l = 0 \tag{15}$$

$$D_2(U_2) \;=\; G\sum_{l=0}^{N} u^2_l W_{N-l} = 0. \tag{16}$$

Thus we have the system $G = 0$ of k equations in k unknowns.

It remains to be shown that (4) is satisfied somewhere. Fix $z \in \mathbf{R^k}$. Let there be $\alpha > 0$ such that for all $r \in \mathbf{R}^2$, $|r| = 1$, we have $r \cdot D_1(p_l) < 0$ for all $y_1 = y_2 = z + \alpha r$, for all l. One must show that we have $r \cdot D_1(U_1) < 0$ and $r \cdot D_2(U_2) < 0$ for all $y_1 = y_2 = z + \alpha r$.

We have

$$r \cdot D_1(U_1) = r \cdot \sum_{l=0}^{N} u^1_l D_1(P_l) = \sum_{i=1}^{N} r \cdot D_1(p_i) V^1_i, \tag{17}$$

where

$$V^1_i = \sum_{l=1}^{N} u^1_l \left\{ \sum_{S \ni i, |S|=l} \left[\prod_{k \in S-\{i\}} p_k \prod_{k \notin S} (1-p_k) \right] - \sum_{S \not\ni i, |S|=l} \left[\prod_{k \notin S, k \neq i} (1-p_k) \prod_{k \in S} p_k \right] \right\} \tag{18}$$

Fix $i = 1$. We have

$$V^1_1 = \sum_{j=1}^{N-1} \left\{ (u^1_{N+1-j} - u^1_{N-j}) \sum_{S \subset N-\{1\}, |S|=N-j} \left(\prod_{k \in S} p_k \prod_{k \notin S} (1-p_k) \right) \right\}. \tag{19}$$

As $u_l^1 \geq u_{l-1}^1$, with strict inequality a least for one l, we have $V_1^1 > 0$. Hence, $r \cdot D_1(U_1) < 0$. Since p_i are smooth functions, the vector field defined by $D_1(U_1(y,y))$ over X does not have singularities. Since all vectors $D_1(U_1(y,y))$ point inside the sphere with the center at z and of radius α, there must be some y^* inside the sphere at which $D_1(U_1(y^*,y^*)) = 0$, and which is a local maximum of $U_1(y_1,y_2)$ with respect to the first argument. If any of the two necessary conditions is satisfied, then we also have $D_2(U_2(y^*,y^*)) = 0$; hence, a local Nash equilibrium.

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